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Wave breakdown in stratified shear flows

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The wave-mechanical condition (Landahl 1972) for breakdown of an unsteady laminar flow into strong small-scale secondary instabilities is applied to some simple stratified inviscid shear flows. The cases considered have one or two discrete density interfaces and simple discontinuous or continuous velocity profiles. A primary wavelike disturbance to such a flow produces a perturbation velocity that is discontinuous at a density interface. The resulting instantaneous system, defined as the sum of the mean flow and the primary oscillation, develops a local secondary shear-flow instability that has a group velocity equal to the arithmetic mean of the instantaneous velocities on the two sides of the interface. According to the breakdown criterion, the disturbed flow will become critical whenever this velocity reaches a value equal to the phase velocity of the primary wave. The calculations show that for a single density interface breakdown may occur for low to moderate wave amplitudes in a fairly narrow range of Richardson numbers on the stable side of the stability boundary. On the other hand, in the unstable regime maximum wave slopes of order unity may be reached before breakdown occurs; this conclusion is in qualitative agreement with experiments. When the system includes two density interfaces, it is found that there exists a range of high Richardson numbers far into the stable regime for which breakdown may take place even for very small and zero wave interface deflexions.

1. Introduction

One of the central problems in the field of turbulence is to understand the mechanism whereby a laminar flow becomes turbulent. It has generally been accepted in the past that transition to turbulence has its origin in flow instabilities, hence the strong interest exhibited in the classical field of hydrodynamic stability theory during the last century. However, it has long since become evident that classical stability theory for a parallel flow is not adequate for explaining the mechanism of transition. First, many laminar flows that are stable to small disturbances, such as pipe flows, do nevertheless become turbulent at high Reynolds numbers. Thus finite amplitude instability would have to be considered. Second, and more important, many shear flows, in particular wall-bounded flows, show a very rapid, almost explosive local growth of eddies of small scales during the final stage of transition. Such rapid growth has been observed in boundary-layer flows by, among others, Klebanoff, Tidstrom & Sargent (1962). Experiments on stratified shear flows by Thorpe (1968, 1969, 1971, 1973) also provide evidence of a rapid onset of small-scale turbulence during breakdown of an instability wave.

The appearance of small-scale disturbances can result from a secondary instability of the perturbed flow field due to a large-scale primary wavelike disturbance. Considering both primary and secondary disturbances to be wave-like, Landahl (1972) used wave kinematics to show that a critical condition for a self-excited secondary wave train would arise whenever its group velocity became equal to the phase velocity of the primary wave. At this condition the secondary wave group would be focused and (for a neutrally stable primary wave) trapped at a given local position on the primary wave where it could attain large amplitudes. Landahl's (1972) calculations for the boundary-layer case, using measured instantaneous velocity profiles from the experiments by Klebanoff *et al.* (1962) to determine the growth-rate and dispersion characteristics of the secondary wave on the basis of the Orr–Sommerfeld problem, gave a position of the first appearance of breakdown and a frequency of secondary waves in good agreement with the experiments. The secondary instability arises because the primary instability wave induces an internal shear layer that produces inviscidly unstable inflexional velocity profiles near the crest of the primary wave.

The possibility of focusing of short waves on long ones has also been noted in connexion with studies of specific wave systems. Gargett & Hughes (1972) showed that short surface waves would interact strongly with long-crested internal waves of large wavelengths whenever the component of the surface-wave group velocity normal to the crest of the internal wave became equal to the phase velocity of the latter, in correspondence with Landahl's (1972) breakdown criterion. Phillips (1973, 1976, chap. 3) has also demonstrated that the $c_g = c_0$ criterion can be derived on basis of resonant wave interaction for a resonant wave triad consisting of a long wave of wavenumber Δk and two short waves of wavenumbers k and $k + \Delta k$. The breakdown criterion is then recovered for the special case when the amplitude of the long wave tends to zero. This correspondence of resonant triad interaction with breakdown has also been pointed out by M. A. S. Ross (private communication).

The present paper reports on an effort to apply Landahl's breakdown condition to inviscid stratified shear flows. A general development of the governing equations, as well as a discussion of some of the immediate consequences that can be expected when the criterion is applied to stratified shear flows, has already been outlined by Criminale (1972). Unlike Landahl's presentation, however, this formulation was viewed from the standpoint of a two-scale stability analysis and no quantitative results were presented.

For small disturbances, Miles (1961) and Howard (1961) have shown that stratified shear flows are always stable if the local Richardson number is everywhere greater than $\frac{1}{4}$. In the atmosphere, though, observations of clear-air turbulence have been reported for conditions in which the local (mean) Richardson number is greater than $\frac{1}{4}$ (see Mather 1969, for example). Such observations may be reconciled with localized parallel-flow stability theory if one takes account of the fact that a neutrally stable wave in a stratified shear flow may induce a local shear layer which can be intense even for small amplitudes if the density gradient is large, thus leading to secondary instability. Strong local density gradients are often observed in the thermocline of the ocean, where the density variation which depth often resembles a series of step functions. Indeed, Woods (1968) has observed localized patches of small-scale turbulence riding on internal waves in the Mediterranean Sea for a situation where the thermocline was found to be divided into relative thick layers of moderate temperature gradient separated by thin sheets of much higher temperature gradient.

In the present investigation the possibility of breakdown in Landahl's (1972) sense is studied for some highly idealized stratified shear flows. Either one or two discrete density interfaces are considered and the velocity is assumed to be either constant in each layer or to vary linearly in the shear layer. Discrete-interface models are often adopted in stratified-flow research for the study of waves of wavelength large compared with the thickness of the density stratification layer. Using linearized theory Phillips (1976, equation 5.6.3) has shown that, for a long internal wave of the lowest mode, the Richardson number due to the flow induced by the wave will become locally less than $\frac{1}{4}$, thus indicating the possible onset of secondary instability, whenever the wave slope $s_0 = k_0\delta_0$ exceeds the limiting value

$$(s_0)_{\mathrm{lim}} \simeq 2\omega_0/N_m \quad (\ll 1), \tag{1}$$

where N_m is the maximum value of the Brunt–Väisälä frequency

$$N(y) = \{-g(d\rho/dy)/\rho\}^{\frac{1}{2}}$$

in the layer and k_0, δ_0 and ω_0 are the wavenumber, amplitude and frequency of the primary wave respectively. He also finds that $(s_0)_{\mathrm{lim}}$ is always small compared with unity for wavelengths large compared with the thickness of the density layer. For the limiting case of vanishing thickness, corresponding to a discrete density interface, $d\rho/dy$ is infinite, and the secondary wave hence becomes unstable for any primary wave amplitude. The wavelength of the most rapidly growing secondary wave will generally be of the order of the thickness of the density layer. Thus the requirement for the validity of the basic theory that the secondary wave causing breakdown must be of much smaller scale than the primary wave is satisfied whenever the density layer's thickness is much smaller than the primary wavelength. For the discrete-interface case the breakdown problem thus reduces to the kinematical one of determining whether the group velocity of the secondary wave can become equal to the primary wave's phase velocity somewhere along the wave. This problem is much simplified because the velocity field induced by the primary wave is discontinuous at the interface, and all unstable waves induced by the discontinuity will propagate with a velocity equal to the arithmetic mean of the velocities on the two sides of the interface. For layers of finite thickness with simple and symmetrical density and shear profiles, stability investigations show that the fastest growing mode also has a phase velocity equal to the mean of the velocities on the two sides of the layer, so that, again, the discrete-interface model provides a realistic approximation to the case of a density layer of small but finite thicknesss. Phillips (1976) used his analysis to explain how breakdown through secondary instability could restrict the amplitude of internal waves and could provide a mechanism for smoothing out density gradients through turbulent mixing. The present analysis may be regarded as complementary to that of Phillips, stressing instead the additional kinematical condition required for the focusing and trapping of secondary waves leading to local concentration of secondary wave energy and hence to a strong breakdown mechanism. When a mean shear is present, Phillips' analysis of the condition leading to secondary instability needs to be modified, but the main conclusion that the discrete-interface model gives the correct representation of the problem for finite thickness in the limit of infinite primary wavelength remains valid. Such simplifications, therefore, facilitate the analysis but are not limitations on the theory.

In the basic theory developed in the present paper, primary and secondary waves are allowed to be swept relative to the flow direction and hence processes dependent on three-dimensionality are incorporated. It is also assumed that the primary wave amplitude is small so that linearized theory can be used to determine the phase velocity of the wave and the associated perturbation velocity field. This requirement was necessitated by the lack of measured (or *in situ*) instantaneous density and velocity profiles. In principle, the breakdown criterion could be computed for finite amplitude primary waves if such experimental information were available, as was done by Landahl (1972).

2. Basic theory

Consider a wave propagating in an inviscid stratified shear flow of steady velocity distribution $U(y)\,\mathbf{i}$, which may or may not be continuous. The model considered is one in which the fluid has a density jump $\Delta\rho_i$ at one or more interfaces located at $y = y_i$. Let the phase velocity of the primary wave, first assumed to be real, be $\mathbf{c}_0 = c_0 \cos\Lambda\mathbf{i} + c_0 \sin\Lambda\mathbf{k}$, where Λ is the sweep angle of the wave as shown in figure 1. The wave will induce a perturbation velocity field

$$\mathbf{q}_0 = u_0\mathbf{i} + v_0\mathbf{j} + w_0\mathbf{k},$$

which will generally be discontinuous at the density interfaces. The induced velocity discontinuity will cause the development of a shear-layer instability wave with a phase and group velocity equal to the arithmetic mean of the velocities on the two sides of the interface, i.e.

$$\mathbf{c} = \mathbf{c}_g = \tfrac{1}{2}(U^+ + U^-)\,\mathbf{i} + \tfrac{1}{2}(\mathbf{q}_0^+ + \mathbf{q}_0^-) \equiv U_m\mathbf{i} + \mathbf{q}_m, \tag{2}$$

where the superscripts + and − indicate values on the upper and lower sides of the interface respectively. That $\mathbf{c}_g$ is independent of wavenumber for the unstable secondary wave leads to considerable simplifications in the present study.

In addition to requiring the secondary wave to be locally unstable, Landahl's (1972) breakdown criterion specifies the kinematic condition

$$c_{gn} = c_{0n}, \tag{3}$$

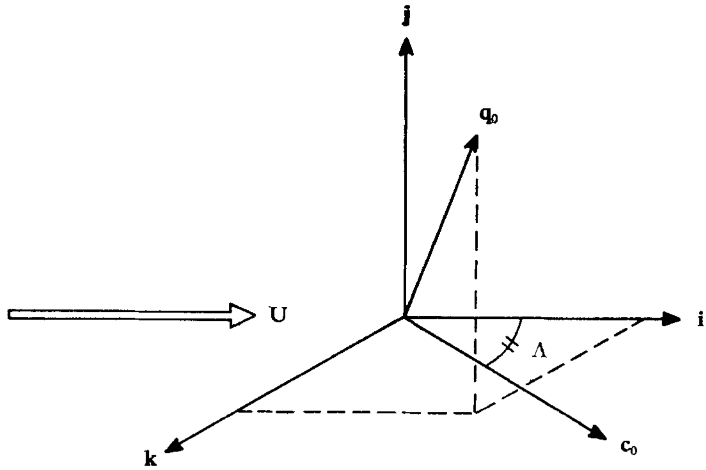

FIGURE 1. Relative orientation of axes illustrating definition of sweep angle Λ, phase velocity and perturbation velocity.

where c_g is the secondary group velocity and the subscript n refers to the components normal to the primary wave front. Projection onto the direction of the wave front gives

$$c_n = \tfrac{1}{2}(U^+ + U^-)\cos\Lambda + \tfrac{1}{2}(q_{0n}^+ + q_{0n}^-) = U_m \cos\Lambda + q_m. \tag{4}$$

The perturbation field for the primary wave and its phase velocity normal to the wave front can be obtained most simply by a rotation of the co-ordinate system through the angle Λ, whence a two-dimensional problem is obtained for a velocity distribution $U_n = U\cos\Lambda$ with an equivalent *overall* Richardson number given by

$$J_{0n} = g\frac{\Delta\rho}{\rho}\frac{d}{(\Delta U)^2\cos^2\Lambda} = \frac{J_0}{\cos^2\Lambda}, \tag{5}$$

where $\Delta\rho$ and ΔU represent reference overall density and velocity differences, respectively, and d is the reference length. Hence the problem of finding the breakdown condition for a swept wave is reduced to an equivalent problem for an unswept one, so that only a two-dimensional problem need be considered. For the case of a neutrally stable primary wave, the solution of this problem is most easily approached in the following fashion. First, a solution for the primary perturbation field satisfying the kinematic boundary conditions at the wavy interfaces for a given interface wavelength and phase velocity c_0 is sought. For the simple flow models considered here this amounts to solving Laplace's equation for the stream function. The phase velocity c_0 is chosen, for a selected value of $u_m = q_m$, such that the breakdown condition (4) is satisfied. The requirement that the pressure must be continuous across the interface gives the value of J_0 required. Finally, the amplitude required to produce the breakdown at any particular location (phase) along the primary wave is computed from the kinematic solution. If there is more than one interface in the problem the requirement that the procedure shall produce the same overall Richardson number for all interfaces gives a compatibility condition between the interface amplitudes which yields the proper eigensolution for the perturbation velocity distribution.

This procedure will generally show that for each wave amplitude the breakdown condition will be satisfied at two positions per primary wavelength, because u_m attains the given value at two different instants per wave cycle, once during the ascending and once during the descending phase of the wave. However, only one of these points will lead to breakdown depending on whether $c_g - c_0$ increases or decreases through the zero. If it increases with x (figure 2a) secondary waves to the left of the zero will travel in the $-x$ direction, i.e. away from the focus. Waves to the right of the focus will travel to the right, i.e. also away from the position where $c_g = c_0$. Hence there can be no accumulation of secondary wave energy at this position. The situation is analogous to that of acceleration of a compressible gas through the speed of sound. For the case where $c_g - c_0$ decreases through zero (analogous to deceleration through the speed of sound), however, the secondary disturbance energy will accumulate at the critical point $c_g - c_0$, leading to breakdown (figure 2b). Thus an auxiliary condition required for breakdown is

$$\partial(c_0 - c_g)/\partial x > 0$$

or, if the primary wave is neutrally stable, simply

$$\partial u_m/\partial x < 0. \tag{6}$$

With the aid of (6), one can thus determine whether the breakdown will occur on the forward or the backward face of the primary wave.

Should the primary wave be growing or decaying, Landahl (1972) showed that an equivalent primary wave velocity c_0' is given by

$$c_0' = -(c_g)_t/(c_g)_x, \tag{7}$$

where the subscripts t and x denote partial derivatives holding the secondary wavenumber constant. The condition for focusing,

$$c_g = c_0', \tag{8}$$

is now different from that of trapping of the secondary wave, namely

$$c_g = c_{0r} \tag{9}$$

with $c_0 = c_{0r} + ic_{0i}$. By considering a temporally growing two-dimensional primary wave with $u_m = a_m \cos\theta_0$, where $\theta_0 = k_0(x - c_{0r}t)$ is the phase measured with respect to the primary wave crest and where $a_m = \hat{u}_m \exp(k_0 c_{0i} t)$, $\hat{u}_m$ being the initial velocity amplitude, one obtains from (7) and (2)

$$c_0' = -(u_m)_t/(u_m)_x = c_{0r} + c_{0i}\cot\theta_0. \tag{10}$$

The breakdown condition (8) with c_g given by (4) leads to

$$U_m + \alpha_m \cos\theta_0 = c_{0r} + c_{0i}\cot\theta_0, \tag{11}$$

which generally will be satisfied at two values of θ_0. Either $\partial(c_0' - c_g)/\partial x < 0$ or $\partial(c_0' - c_g)/\partial x > 0$, the latter being the critical condition. The minimum amplitude

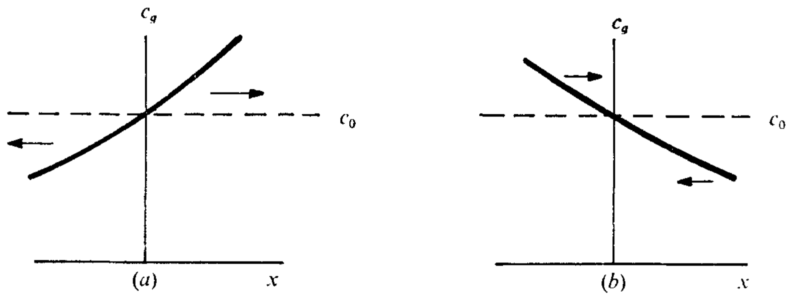

FIGURE 2. Conceptual figure illustrating direction of energy propagation relative to a focus. (*a*) Non-critical. (*b*) Critical.

a_m for breakdown is obtained when $\partial(c_g - c_0')/\partial x = 0$ at the position where (11) is satisfied, i.e. at the phase given by

$$\sin\theta_0 = (c_{0i}/a_m)^{\frac{1}{3}}. \tag{12}$$

By inserting (12) into (11) and solving for a_m we obtain

$$a_m = [(c_{0r} - U_m)^{\frac{2}{3}} + c_{0i}^{\frac{2}{3}}]^{\frac{3}{2}} \tag{13}$$

as the minimum amplitude required for breakdown. It follows from this that trapping of the secondary wave, occurring when

$$u_m = c_{0r} - U_m, \tag{14}$$

always requires a smaller amplitude than focusing and will first appear at the phase position of maximum u_m (usually at the crest or trough), which will generally be different from that for focusing, except when $c_{0r} - U_m = 0$, in which case focusing and trapping both occur at the phase of zero u_m.

To calculate the primary wave amplitude leading to focusing and hence breakdown for a temporally growing wave, one must first solve for $c_0 = c_{0r} + ic_{0i}$ for given k_0 and J_0, then determine from (13) the induced velocity amplitude a_m required for breakdown, and finally calculate from the kinematic relationship between the primary wave amplitude δ_0 and a_m as a function of k_0 and c_0 the amplitude δ_0 required.

3. Flows with one density interface

To illustrate the method we first consider one density interface with a velocity discontinuity, i.e. the simple Kelvin–Helmholtz problem. The problem is non-dimensionalized by setting the velocity equal to ± 1 on each side of the interface. The density is $\rho^- = \rho + \frac{1}{2}\Delta p$ below and $\rho^+ = \rho - \frac{1}{2}\Delta p$ above the interface, where $\Delta\rho/\rho \ll 1$.

In non-dimensional variables with g made dimensionless through division by the square of the reference velocity $U = 1$ and multiplication by the reference length $d = 1$, the quantity $g\Delta\rho/\rho$ is then equal to J_0. (In this problem there is no natural reference length as is reflected by the fact that J_0 always appears in the combination J_0/k_0, which is independent of the reference length is chosen as long as the same reference length is used for both k_0 and J_0.) The stream function corresponding to an interface displacement of

$$y \equiv \eta_0 = \hat{\delta}_0 \exp[ik_0(x-c_0 t)] \tag{15}$$

is given by

$$\psi^{\pm} = -\hat{\delta}_0(\pm 1 - c_0)\exp[ik_0(x-c_0 t) \mp |k_0|\, y], \tag{16}$$

where the upper sign refers to the region above the interface ($y > 0$) and the lower sign to the region below ($y < 0$).

Without loss of generality we may in the following assume the primary wave-number k_0 to be positive. The induced velocity $u_0^{\mp}$ is given by $\psi_y^{\pm}$, and hence from (16)

$$u_m = k_0\hat{\delta}_0 \exp[ik_0(x-c_0 t)] = k_0\eta_0, \tag{17}$$

so that for breakdown at a given η_0 we should have, according to (4),

$$c_0 = k_0\eta_0. \tag{18}$$

Bernoulli's equation gives for the perturbation pressure

$$p^{\pm}/\rho^{\pm} = -\psi_y^{\pm}(\pm 1 - c_0) - g\eta^{\pm}, \tag{19}$$

where η is the vertical displacement of the fluid particle. Hence, assuming $\Delta\rho/\rho \ll 1$, we find that the continuity of pressure across the interfaces requires

$$\begin{aligned} g\Delta\rho/\rho \equiv J_0 &= [\psi_y^-(1+c_0) + \psi_y^+(1-c_0)]_{y=0}/\eta_0 \\ &= 2k_0(1+k_0^2\eta_0^2). \end{aligned} \tag{20}$$

The primary wave will become unstable for $J_0 < 2k_0$. According to (20), a wave of small but finite amplitude will suffer breakdown just before the instability sets in. The intensity of the secondary instability generated depends on the induced velocity jump across the interface, which is given by

$$\Delta u_0 = u_0^- - u_0^+ = 2k_0 c_0 \eta_0, \tag{21}$$

which at breakdown, following (18), is small, i.e.

$$\Delta u_0 = 2k_0^2\eta_0^2. \tag{22}$$

According to (6) and (17) this weak breakdown should appear on the part of the wave where $\partial\eta/\partial x < 0$, i.e. on the forward face of the wave (assuming the wave to travel to the right) with $\eta_0 > 0$.

For the unstable regime $J_0 < 2k_0$ one has

$$c_{0r} = 0, \quad c_{0i} = (1 - J_0/2k_0)^{\frac{1}{2}}. \tag{23}$$

From (13) and (16) it then follows that the minimum wave amplitude for break-down of an unstable Kelvin–Helmholtz wave is given by

$$k_0\delta_0 = 1(-J_0/2k_0)^{\frac{1}{2}} \tag{24}$$

with $\delta_0 = \hat{\delta}_0 \exp(k_0 c_{0i} t)$. The maximum primary wave slope $k_0\delta_0$ attainable before breakdown sets in is thus unity for $J_0 = 0$. The linearized theory is, of course, only of qualitative value at best for such high wave amplitudes.

A more realistic model for a free shear layer with a density interface is that investigated by Holmboe (1962). This model consists of a layer of constant shear

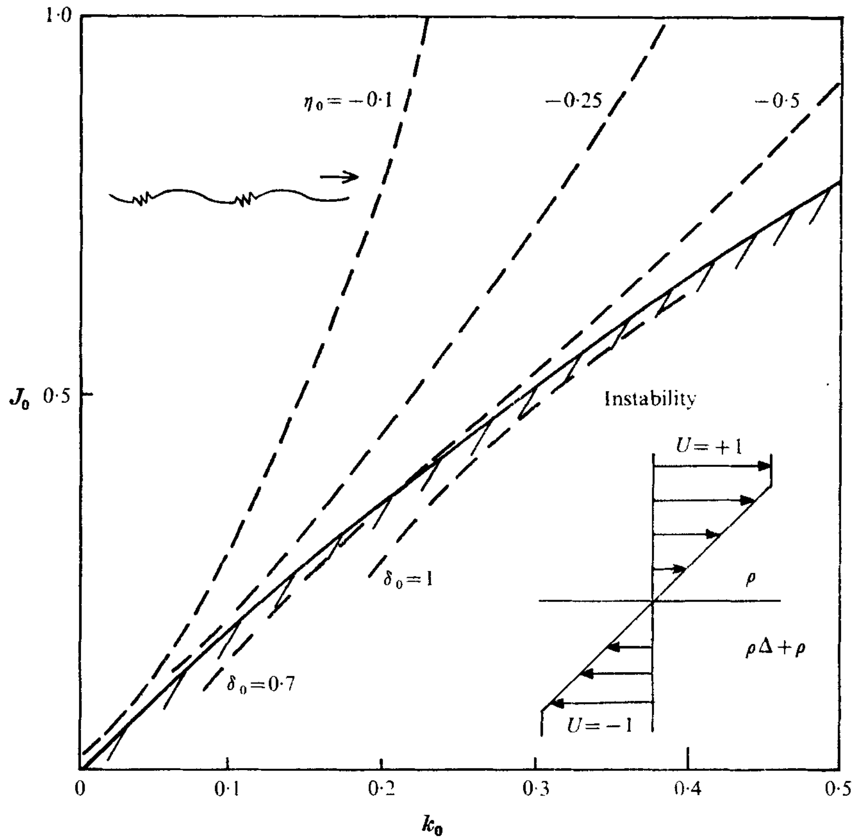

FIGURE 3. Results of breakdown calculations for the Holmboe (1962) model of a stratified shear layer. Insert illustrates on which face of the primary wave breakdown will occur.

of finite thickness with the density interface located in the middle, as illustrated in figure 3. The non-dimensional thickness is taken to be 2 with the velocity on each side of the layer ± 1 (as in the Kelvin–Helmholtz model). The kinematical solution is given by

$$\psi^{\pm} = \begin{cases} A_0^{\pm} \exp\left[ik_0(x - c_0 t) - k_0 |y|\right] & \text{for} \quad |y| \geqslant 1, & (25) \\ (A_1^{\pm} e^{\pm k_0 y} + B_1^{\mp} e^{\mp k_0 y}) \exp\left[ik_0(x - c_0 t)\right] & \text{for} \quad -1 \leqslant y \leqslant +1, & (26) \end{cases}$$

with
$$\left.\begin{aligned} A_0^{\pm} &= 2k_0 B_1^{\pm}(1 \mp c_0)\, e^{2k_0}, \\ A_1^{\pm} &= B_1^{\pm}[2k_0(1 \mp c_0) - 1]\, e^{2k_0} \\ B_1^{\pm} &= c_0 \hat{\eta}\, e^{-2k_0}/(G \mp 2k_0 c_0), \end{aligned}\right\} \qquad (27)$$
and

where
$$G = 2k_0 - 1 + e^{-2k_0}.$$

From this solution one finds that at the interface $y = 0$

$$u_m = \frac{\xi\eta}{G^2 - \xi} e^{-2k_0}, \qquad (28)$$

where
$$\xi = 4k_0^2 c_0^2.$$

Continuity of pressure at $y = 0$ requires

$$J_0 = \frac{\xi}{2k_0}\left(1 - \frac{2G\,e^{-2k_0}}{G^2-\xi}\right). \tag{29}$$

By solving (29) for ξ, and hence for c_0, one finds that instability of a non-travelling ($c_{0r} = 0$) wave mode, similar to the Kelvin–Helmholtz mode, is possible for $\exp(-2k_0) - 2k_0 + 1 > 0$, or $k_0 \lesssim 0{\cdot}639$ whenever $J_0 < J_{01}$, where

$$J_{01} = (2k_0)^{-1}\,[G - (2G\,e^{-2k_0})^{\frac{1}{2}}]^2. \tag{30}$$

A second travelling-wave unstable mode exists for $J_{01} < J_0 < J_{02}$, where

$$J_{02} = (2k_0)^{-1}\,[G + (2G\,e^{-2k_0})^{\frac{1}{2}}]^2, \tag{31}$$

and can occur for all values of k_0.

On the basis of (28) and (29) the interface deflexion leading to breakdown was calculated for neutrally stable waves travelling to the right ($c_0 > 0$) and with values of J_0 near the upper stability boundary (31). The results are shown in figure 3. As may be seen, fairly moderate deflexions are needed for low values of k_0, and breakdown will first occur in the troughs and on the backward faces of the wave. For the wave travelling to the left ($c_0 < 0$) breakdown will occur on the crests.

Interface amplitudes required for breakdown in the unstable regime near the upper stability boundary have also been determined and are included in figure 3. As may be seen, the amplitudes rapidly become large inside the unstable regime as one departs from the immediate neighbourhood of the stability boundary. Inside the region of first-mode instability the breakdown amplitudes found are qualitatively those that are found for the Kelvin–Helmholtz problem. The maximum wave slope attainable before breakdown is found for the most unstable wave with $J_0 = 0$ and $k_0 \simeq 0{\cdot}398$ to be $k_0\delta_0 \simeq 0{\cdot}615$.

Experiments on a shear flow with a rather sharp density interface have been carried out by Browand and coworkers (Browand & Wang 1972; Browand & Winant 1973; see Maxworthy & Browand 1975), who have observed breakdown both for mode 1 and mode 2 waves. Breakdown of the former was found to occur in much the same manner as that observed by Thorpe. For mode 2 waves breakdown was observed to be confined to the high-speed side of the flow and would be consistent with the present model if the wave breaking were the one travelling upstream with respect to the mean velocity ($c_0 < 0$ in the present model). For growing waves this is the one that may be expected to amplify in the shortest distance downstream since its group velocity, measured in the laboratory frame, would be the smallest. In the experiments the breaking mode 2 waves were formed by a collapse of a mode 1 wave producing two sets of waves (Browand, private communication), one travelling upstream and one downstream with respect to the mean speed, but no information is available as to which set ultimately broke. The experimental amplitudes at breakdown were moderately large, thus lending qualitative support to the breakdown theory.

The effect of a bounding rigid surface may be studied by considering an example that represents an idealized model of a cold bottom current (see

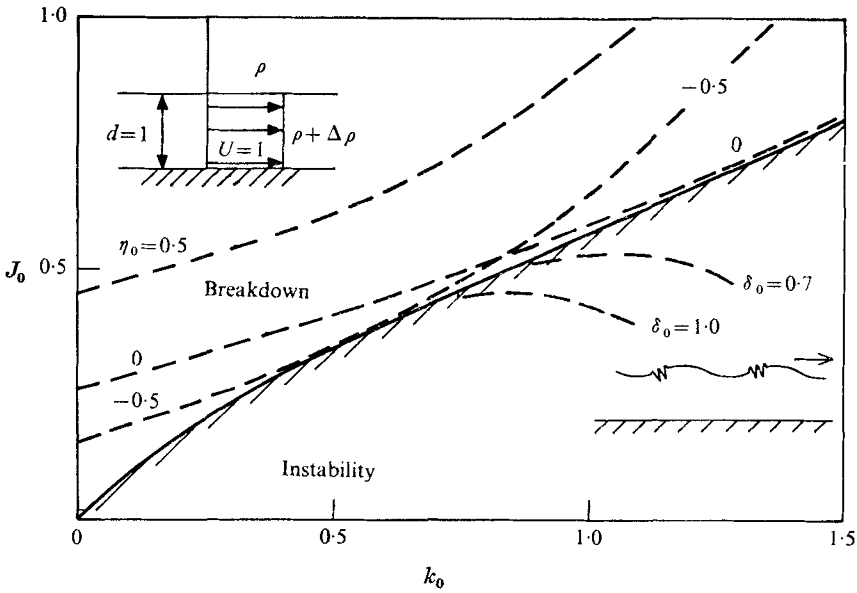

FIGURE 4. Results of breakdown calculations for waves on a cold bottom current. Insert illustrates on which face of the primary wave breakdown will occur.

figure 4). The depth d is set equal to unity in the non-dimensionalized problem. The solution for the stream function for given c_0 satisfying the kinematic boundary conditions at $y = 0$ and $y = 1$ is easily found to be

$$\psi^- = -(1-c_0)\,\hat{\delta}_0 \frac{\sinh(k_0 y)}{\sinh k_0} \exp[ik_0(x-c_0t)] \tag{32}$$

and

$$\psi^+ = c_0\,\hat{\delta}_0 \exp[ik_0(x-c_0t) - k_0(y-1)]. \tag{33}$$

From this we find

$$u_m = -\tfrac{1}{2}k_0[(1-c_0)\coth k_0 + c_0]\,\eta_0 \tag{34}$$

and from the condition (4) for breakdown

$$c_0 = \tfrac{1}{2} - \tfrac{1}{2}k_0[(1-c_0)\coth k_0 + c_0]\,\eta_0,$$

i.e.

$$c_0 = \frac{1-k_0\eta_0\coth k_0}{2+k_0\eta_0(1-\coth k_0)}. \tag{35}$$

The requirement that the pressure be continuous across the interface then gives, assuming $(\rho^+-\rho^-)/(\rho^++\rho^-) \ll 1$ as before,

$$g\Delta\rho/\rho \equiv J_0 = -[c_0\psi_y^+ + \psi_y^-(1-c_0)]_{y=1}/\eta,$$

or after substitution of (31) and (32),

$$J_0 = k_0[c_0^2 + (1-c_0)^2\coth k_0]. \tag{36}$$

By introducing (35) into (34) one finds

$$u_m = -\frac{k_0\eta_0}{2}\frac{1+\coth k_0}{[2+k_0\eta_0(1-\coth k_0)]}, \tag{37}$$

which shows that u_m and η_0 have opposite signs. Hence breakdown will occur on portions of the wave for which $\partial\eta_0/\partial x > 0$, i.e. on the backward face.

Sample results given by (35) and (36) are shown in figure 4. Also indicated is the region of instability for the primary wave. It may be seen that there exists a region of neutrally stable primary waves for which breakdown can already occur at small interface deflexions. For low primary wavenumbers this region extends only to moderately high Richardson numbers. In the unstable regime focusing will require high amplitudes at low wavenumbers ($k_0 < 0{\cdot}8$) as seen in figure 4, in which some amplitudes for breakdown have been calculated from (13) and (34). In the range of higher wavenumbers the amplitudes become smaller, and in the neighbourhood of the stability boundary, breakdown may occur at fairly low amplitudes. Trapping in this case takes place at a different position on the wave from focusing, except in the limit $k_0 \to \infty$, for which $c_{0r} \to U_m$.

4. Flow with two interfaces

As a simple example of a stratified shear flow with two density interfaces and with a continuous shear-flow profile we consider the example shown in figure 5. This is a model that has been investigated for instability by Taylor (1932) and Goldstein (1932). As in the Holmboe model the velocity is assumed to vary linearly between $U = -1$ at the lower interface, located at $y = -1$, and $U = +1$ at the upper, located at $y = 1$. The density of the middle layer is assumed to lie halfway between those of the upper and lower layers and the total density difference is taken to be $2\Delta\rho$. As before it is assumed that $\Delta\rho/\rho \ll 1$.

Let the upper, middle and lower layers be denoted by superscripts $+$, m and $-$ respectively. Also, let the locations of the perturbed upper and lower interfaces given by

$$y^+ = 1+\hat{\delta}_0^+ \exp[ik_0(x-c_0t)], \quad y^- = 1+\hat{\delta}_0^- \exp[ik_0(x-c_0t)], \tag{38}$$

respectively. The linearized solutions for the stream function due to the interface perturbations satisfying the kinematical boundary conditions at $y = \pm 1$ are, in the different regions,

$$\psi^+ = -(1-c_0)\,\hat{\delta}_0^+ \exp[-k_0(y-1)+ik_0(x-c_0t)], \tag{39}$$

$$\psi^m = [A\,e^{-k_0y}+B\,e^{k_0y}]\exp[ik_0(x-c_0t)], \tag{40}$$

$$\psi^- = (1+c_0)\,\hat{\delta}_0^- \exp[k_0(1+y)+ik_0(x-c_0t)], \tag{41}$$

where

$$A = \frac{(1-c_0)\,\hat{\delta}_0^+\,e^{-k_0}+(1+c_0)\,\hat{\delta}_0^-\,e^{k_0}}{e^{2k_0}-e^{-2k_0}} \tag{42}$$

and

$$B = -\frac{(1-c_0)\,\hat{\delta}_0^+\,e^{k_0}+(1+c_0)\,\hat{\delta}_0^-\,e^{-k_0}}{e^{2k_0}-e^{-2k_0}}. \tag{43}$$

The pressure in an inviscid parallel shear flow $U(y)$ perturbed by a wavelike disturbance is given by

$$p = -\rho(U-c_0)\,\psi_y+\rho\psi U-\rho g y. \tag{44}$$

Combination of (39)–(44) shows that continuity of pressure at the upper interface requires

$$J_0 = k_0(1-c_0)^2+(1-c_0)\,[k_0(1-c_0)\cosh(2k_0)-1]+\frac{\hat{\delta}_0^-}{\hat{\delta}_0^+}\frac{(1-c_0^2)}{\sinh(2k_0)}k_0, \tag{45}$$

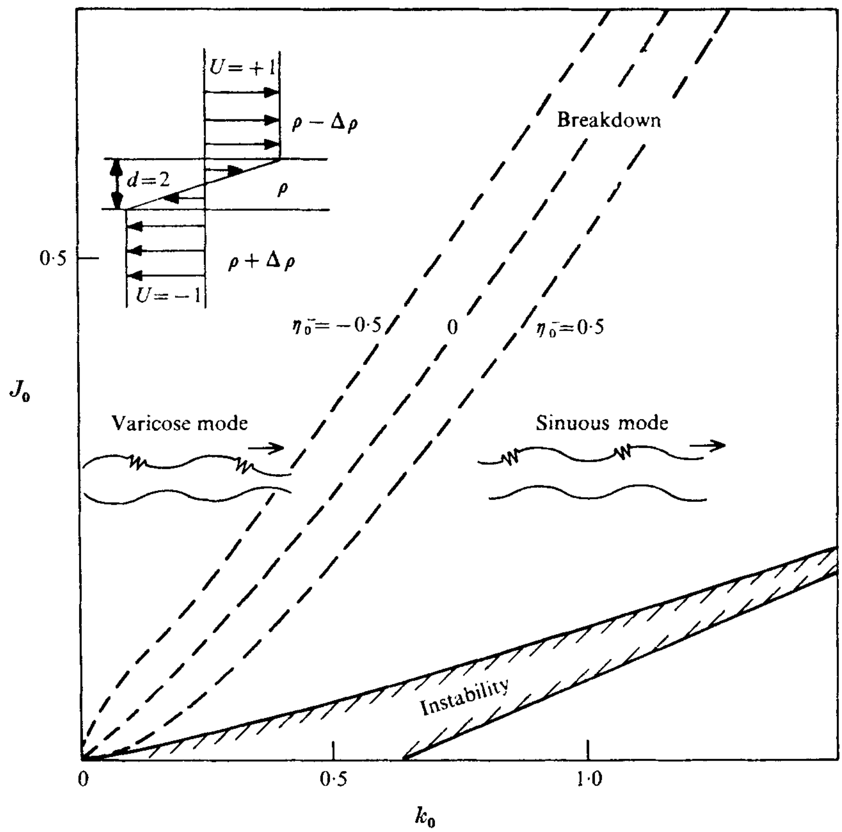

FIGURE 5. Results of breakdown calculations for waves on a two-layer shear flow. Inserts illustrate location of breakdown on primary wave for varicose and sinuous modes.

whereas continuity of pressure at the lower interface gives

$$J_0 = k_0(1+c_0)^2 + (1+c_0)\,[k_0(1+c_0)\cosh 2k_0 - 1] + \frac{\hat{\delta}_0^+}{\hat{\delta}_0^-}\frac{(1-c_0^2)}{\sinh(2k_0)}\,k_0. \tag{46}$$

Compatibility of the two values gives the following requirement for the ratio of the amplitudes:

$$\frac{\hat{\delta}_0^+}{\hat{\delta}_0^-} - \frac{\hat{\delta}_0^-}{\hat{\delta}_0^+} = P, \tag{47}$$

where

$$P = \frac{2c_0 \sinh 2k_0}{k_0(c_0^2-1)}\,[2k_0\cosh 2k_0 + 2k_0 - 1]. \tag{48}$$

Thus

$$\hat{\delta}_0^+/\hat{\delta}_0^- = \tfrac{1}{2}P \pm (\tfrac{1}{4}P^2+1)^{\frac{1}{2}}. \tag{49}$$

The two signs in (49) represent two different modes: a varicose one with $\hat{\delta}_0^+/\hat{\delta}_0^- < 0$, i.e. with the two interfaces deflected in opposite directions, and a sinuous one having $\hat{\delta}_0^+/\hat{\delta}_0^- > 0$, i.e. both interfaces deflected in the same direction. For breakdown at the upper interface we require

$$c_0 = 1 + u_m^+, \tag{50}$$

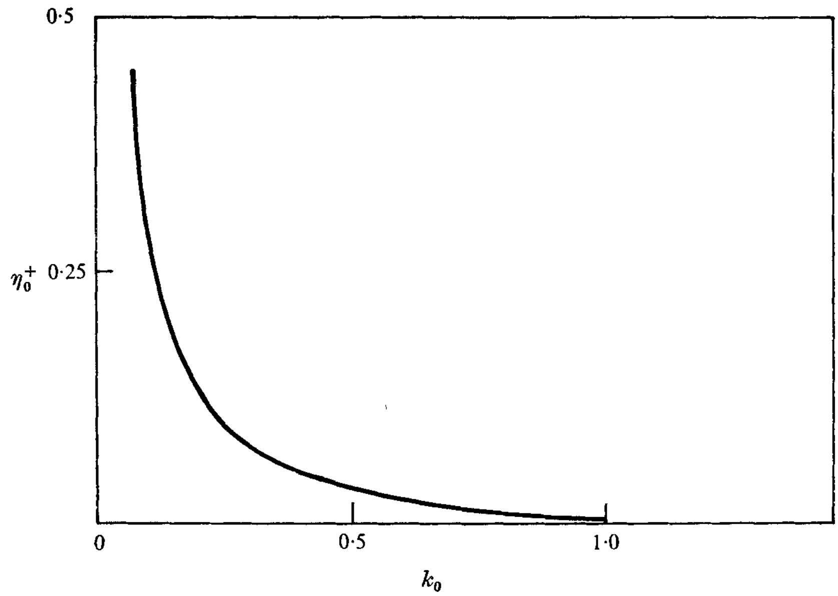

FIGURE 6. Deflexion of upper interface at breakdown in the two-layer model for a (non-dimensional) deflexion of the lower interface of $|\eta_0^-| = 0{\cdot}5$.

hence $c_0 - 1 \ll 1$, and it follows from a comparison of (45) and (46) that $\hat{\delta}_0^+/\hat{\delta}_0^-$ must approach zero in the limit $u_m^+ \to 0$. Therefore, when breakdown occurs at the upper interface this is only slightly deflected (except in the very low wavenumber regime) in comparison with the lower one.

From the kinematic solution we obtain

$$u_m^+ = -\frac{k_0}{2}\left\{(1-c_0)\left[\cosh(2k_0)-1\right]\eta_0^+ + \frac{1+c_0}{\sinh(2k_0)}\eta_0^-\right\}, \tag{51}$$

in which the first term may thus be neglected at breakdown compared with the second. Further, to first order in u_m^+ we may replace c_0 by unity, so that

$$u_m^+ \simeq -k_0\eta_0^-/\sinh(2k_0), \tag{52}$$

which may be used to estimate c_0 from (42). This value is then inserted into (45) (in which the last term may be neglected), giving the Richardson number leading to breakdown. Results of such calculations for $\eta_0^- = -0{\cdot}5$, 0 and $+0{\cdot}5$ are shown in figure 5.† From (47) and (48), we find that

$$\frac{\hat{\delta}_0^+}{\hat{\delta}_0^-}\frac{\eta_0^+}{\eta_0^-} \simeq -P^{-1} = -\frac{k_0 u_m^+}{\sinh(2k_0)\left[2k_0\cosh(2k_0)+2k_0-1\right]}, \tag{53}$$

which in combination with (46) gives

$$\eta_0^+ \simeq \frac{k_0^2(\eta_0^-)^2}{\sinh(2k_0)}\frac{1}{\left[2k_0\cosh(2k_0)+2k_0-1\right]}, \tag{54}$$

† In interpreting the results for $-\eta_0 = 0$, it should be remembered that η_0 denotes the interface deflexion and not the wave amplitude, which must be greater than the value of η_0 at breakdown in order for accumulation of secondary wave energy to take place.

showing that for a neutrally stable wave breakdown always occurs on the crest portion of the wave on the upper interface. From (52) it follows that $\partial u_m^+/\partial x$ has the opposite sign to $\partial \eta_0^-/\partial x$. Thus, according to (6) breakdown takes place on that phase of the wave for which η_0^- increases with x. Figure 6 shows the deflexion of the upper interface at breakdown for $|\eta_0^-| = 0{\cdot}5$, calculated from (52). As may be seen, the upper interface is only very slightly deflected under conditions for breakdown.

No calculations have been carried out for the unstable regimes for this model.

5. Conclusions

The simple models of stratified shear flow having discrete density interfaces examined with the aid of linearized theory indicate that wave breakdown due to focusing and trapping of secondary instabilities may occur for certain ranges of Richardson numbers inside the stable regime. For the simple Kelvin–Helmholtz model this range has a width given by $\Delta J_0/2k_0 = k_0^2\eta_0^2$, showing that waves near the stability boundary would break at small wave amplitudes. For a temporally growing Kelvin–Helmholtz wave breakdown occurs first at the point of zero interface deflexion and for a maximum wave slope of $k_0\delta_0 = (1-J_0/2k_0)^{\frac{1}{2}}$.

The more refined Holmboe model, having a shear layer of finite width, was found to be susceptible to breakdown at moderately low Richardson numbers near the upper stability boundary for the second (travelling) mode. Waves well inside the instability region can reach fairly high amplitudes before breakdown, the situation inside the instability region of the first (non-travelling) mode being qualitatively similar to the Kelvin–Helmholtz case.

The idealized model for a cold bottom current also showed a range of Richardson numbers in the stable regime for which breakdown is possible at fairly low wave amplitudes. Inside the unstable regime breakdown due to focusing may occur at moderately large amplitudes for high wavenumbers.

In the case of a simple shear flow having two density interfaces (the Taylor–Goldstein model) the calculations showed that breakdown is possible at zero interface deflexion for high Richardson numbers far inside the stable regime. Both sinuous and varicose modes can suffer breakdown; the interface on which the breakdown occurs is found to have a very small deflexion compared with the other one, except at low wavenumbers.

At present there are not sufficiently detailed experimental results available for a quantitative comparison with the theory. Woods' (1968) field observations in the Mediterranean pertained mainly to what in the present theory would be the secondary instability wave. His estimates of the mean and wave-induced local shear indicated that the local Richardson number was below 0·25 as required by the parallel-flow theory for instability. However, the propagation characteristics of the large-scale (primary) wave inducing the local instability were not determined, so that his data cannot be used for a test of the kinematical breakdown condition. Thorpe's (1968, 1969, 1971, 1973) experiments on temporally growing waves gave wave slopes for the first appearance of breakdown of 1·2–1·4 at the lowest Richardson numbers, with decreasing values, approaching zero at the stability boundary, for increasing Richardson numbers. This is in qualitative agreement with the present theory; see (24). The visual observations of miscible fluids indicated, however, that wave breakdown was preceded by local overturning, so that gravitational collapse would certainly be involved in the mechanism. On the other hand, his (1969) experiments with immiscible fluids, which would be more relevant for a comparison with the theory, show that small-scale irregularities first appear before overturning takes place. The value of 1·2 reported by Thorpe (1969) as the wave slope for the first appearance of these irregularities is in reasonable qualitative agreement with the present simplified theory.

The experiments by Browand & Winant (1973) are of particular interest for the present theory in that they show the appearance of a 'one-sided' breakdown confined to the high-speed side of the wave. The theory indeed shows such a one-sidedness. This should become progressively more marked as the Richardson number increases from inside the region of the first-mode instability, in which the breakdown is symmetrical, to the stability boundary of the second mode, at which the breakdown will first occur at a crest or trough depending on the direction of travel of the wave relative to the mean flow.

The theory presented may have relevance to the mechanism of localized turbulence production in a stratified ocean and to the phenomenon of clear-air turbulence. In particular, breakdown could be expected to limit the wave amplitudes attainable in a stratified medium, so that turbulence spectra for the atmosphere and the ocean could be expected to be depleted in wavenumber regimes for which breakdown starts at low amplitudes. Observations to check such aspects of the theory are not available at present, and it would be necessary to await more experimental data as well as theoretical calculations for more realistic stratified shear flows.

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